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STRESS ANALYSIS OF BRIDGE DECKS

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Abstract—The direct determination of the bending and twisting moments, and shear forces, in orthotropic bridge decks is considered in this paper. The partial differential equation of plate theory is reduced to a set of ordinary linear differential equations, in sets of two, each set containing two unknown functions only, by the assumption that the stress resultants may be expressed as Fourier series in the spanwise co-ordinate, the coefficients of the series being functions of the transverse position only. The series are chosen to satisfy the equilibrium conditions for the plate, the unknown coefficients being obtained by the Principle of Least Work.

NOTATION

$0(x, y)$	co-ordinate axes
η, ξ	non-dimensional co-ordinates
l	span of bridge slab
c	semi-chord of slab
t	thickness of slab
r_i	$l/\pi ic$
w	deflection of middle surface
E_x, E_y, E_{xy}, G_{xy}	elastic moduli defining stress–strain relationships in orthotropic plates
D_x, D_y, D_1, D_{xy}	flexural and torsional rigidities for orthotropic plate
A_1, A_2, A_{12}, A_3	strain energy coefficients
M_x, M_y, M_{xy}	bending and twisting moments per unit length
S_x, S_y	shear forces per unit length of plate
p	intensity of applied load
D	operator $d/d\xi$
S, F_i, R_i	moment functions

INTRODUCTION

IN THE analysis and design of modern highway bridges, it has become necessary to consider the effects of the heavy indivisible loads which are using the British road system in increasing numbers. In order to achieve economic designs, an accurate assessment of the load distribution in the bridge deck, due to such concentrated loads, is essential.

The analysis is generally carried out by replacing the physical composite slab by an equivalent orthotropic plate or grid structure, which may then be treated by conventional analytical methods. In the approach developed by Massonnet [1], a solution is obtained by assuming that the deflected form of the plate may be represented by a one-dimensional Fourier series; by expressing the applied load as a similar series, the coefficients are obtained by equating corresponding terms in the governing biharmonic equation. The moments and shear forces are determined subsequently by double and triple differentiation of the deflection function. However, it is well known that when deflections are approximated, the stresses calculated by differentiation of the deflection function are in greater error than the corresponding deflections.

The purpose of the present paper is to illustrate a technique whereby the moments and shear forces may be determined directly. The method is similar to Massonnet's approach, except that assumed distributions of moments and shear forces are used instead of an assumed deflection function.

The bending and twisting moments, and shear forces, are expressed as Fourier series in the spanwise co-ordinate, the coefficients of the series being functions of the transverse position only. The series are chosen to satisfy both the equilibrium equations and boundary conditions for the plate, and the unknown coefficients are determined by minimization of the strain energy. Considerable simplification is achieved by splitting the applied load system into symmetrical and anti-symmetrical components.

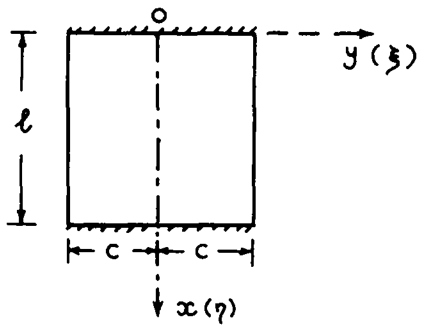

FIG. 1. Bridge slab.

ANALYSIS

The structure considered is a thin elastic orthotropic bridge slab of uniform thickness, simply supported on two opposite edges (Fig. 1).

For convenience, a set of non-dimensional co-ordinates, (η, ξ), is used, defined with reference to Fig. 1 as

$$\eta = \frac{\pi x}{l}, \qquad \xi = \frac{y}{c} \tag{1}$$

The stress resultants in the plate must obey the usual "small-deflection" equations of equilibrium, which become, in terms of the non-dimensional co-ordinate system,

$$\left.\begin{aligned} \frac{\pi}{l}\frac{\partial M_x}{\partial \eta}-\frac{1}{c}\frac{\partial M_{xy}}{\partial \xi}-S_x &= 0 \\ \frac{1}{c}\frac{\partial M_y}{\partial \xi}-\frac{\pi}{l}\frac{\partial M_{xy}}{\partial \eta}-S_y &= 0 \\ \frac{\pi}{l}\frac{\partial S_x}{\partial \eta}+\frac{1}{c}\frac{\partial S_y}{\partial \xi}+p &= 0 \end{aligned}\right\} \tag{2}$$

where M_x, M_y and M_{xy} are the bending and twisting moments respectively, S_x and S_y are the shear forces, all per unit length of plate, and p is the intensity of applied loads.

If shear deformations are considered in the analysis, the physically correct boundary conditions of Reissner [2] may be utilized. These become

(i) along the free edges, $\xi = \pm 1$,

$$M_y = M_{xy} = S_y = 0 \tag{3}$$

(ii) along the simply supported edges, $\eta = 0, \pi$,

$$w = M_x = M_{xy} = 0 \tag{4}$$

where w is the deflection of the middle-surface of the slab.

If shear deformations are neglected, the usual conditions of Kirchhoff [2] must be used; these are

(i) along the free edge, $\xi = \pm 1$,

$$M_y = S_y - \frac{\pi}{l}\frac{\partial M_{xy}}{\partial \eta} = 0 \tag{5}$$

(ii) along the simply-supported edges, $\eta = 0, \pi$,

$$M_x = w = 0. \tag{6}$$

The general moment–curvature relationships for an orthotropic slab may be written as [2]

$$M_x = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right) \tag{7a}$$

$$M_y = -\left(D_1 \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right) \tag{7b}$$

$$M_{xy} = 2D_{xy} \frac{\partial^2 w}{\partial x \partial y} \tag{7c}$$

where D_x, D_y, D_1 and D_{xy} are the flexural and torsional rigidities for an orthotropic plate.

In this case, as w vanishes along the supported edges, the transverse curvature $\partial^2 w/\partial y^2$ is zero, and thus, from equation (7a), $\partial^2 w/\partial x^2$ is also zero if the normal moment M_x is to vanish. Hence, from equation (7b), the transverse bending moment M_y must also vanish along a supported edge, and this may be used as an alternative boundary condition to a vanishing deflection along the edges $x = 0, b$. This alternative condition is more appropriate to the present analysis.

Any applied load system on the plate may always be expressed as a Fourier series of the form

$$p = \sum_{i=1}^{\infty} p_i \sin i\eta \tag{8}$$

where p_i is a function of the transverse co-ordinate ξ only.

By expressing the load system in this form, statically correct solutions to the equilibrium equations (2), which also satisfy the boundary conditions along the supported edges of the slab, will obviously be obtained if the stress resultants are expressed as corresponding Fourier series of the form

$$
\left.\begin{aligned}
M_x &= \sum_{i=1}^{\infty} M_{x_i} \sin i\eta \\
M_y &= \sum_{i=1}^{\infty} M_{y_i} \sin i\eta \\
M_{xy} &= M_{xy_0} + \sum_{i=1}^{\infty} M_{xy_i} \cos i\eta \\
S_x &= S_{x_0} + \sum_{i=1}^{\infty} S_{x_i} \cos i\eta \\
S_y &= \sum_{i=1}^{\infty} S_{y_i} \sin i\eta
\end{aligned}\right\} \tag{9}
$$

in which the coefficients of the series are again functions of the transverse co-ordinate ξ only.

Equations (9) satisfy Kirchhoff's conditions. If shear deformations are considered, satisfaction of the appropriate boundary conditions (4) will be obtained if M_{xy_0} is expressed in terms of the functions M_{xy_i}. Initially, shear deformations will be neglected, and Kirchhoff's boundary conditions are used.

Substitution of equations (8) and (9) into the equilibrium conditions (2) yields three equations which must be true for every spanwise position on the slab. Hence, equating coefficients of corresponding terms, and solving the resulting set of equations, the stress resultants may be expressed in terms of chosen functional coefficients as

$$
\left.\begin{aligned}
M_x &= \sum_{i=1}^{\infty} \left\{ r_i^2 \frac{d^2 F_i}{d\xi^2} + 2r_i \frac{dR_i}{d\xi} + p_i c^2 r_i^2 \right\} \sin i\eta \\
M_y &= \sum_{i=1}^{\infty} F_i \sin i\eta \\
M_{xy} &= \mathrm{S} + \sum_{i=1}^{\infty} R_i \cos i\eta \\
S_x &= \sum_{i=1}^{\infty} \frac{1}{cr_i} \left\{ r_i^2 \frac{d^2 F_i}{d\xi^2} + r_i \frac{dR_i}{d\xi} + p_i c^2 r_i^2 \right\} \cos i\eta - \frac{1}{c} \frac{d\mathrm{S}}{d\xi} \\
S_y &= \sum_{i=1}^{\infty} \frac{1}{cr_i} \left\{ r_i \frac{dF_i}{d\xi} + R_i \right\} \sin i\eta
\end{aligned}\right\} \tag{10}
$$

where $r_i = l/\pi ic$, and the unknown functional coefficients are defined as

$$\mathrm{F}_i = M_{yi}, \qquad \mathrm{R}_i = M_{xyi}, \qquad \mathrm{S} = M_{xyo}.$$

The stress–strain relationships for an orthotropic plate may be expressed in the form

$$
\begin{aligned}
\sigma_x &= E_x e_x + E_{xy} e_y \\
\sigma_y &= E_{xy} e_x + E_y e_y \\
\tau_{xy} &= G_{xy} \gamma_{xy}
\end{aligned}
$$

where σ_x, σ_y and τ_{xy} are the direct and shear stresses, e_x, e_y and γ_{xy} are the corresponding strains, and E_x, E_y, E_{xy} and G_{xy} are elastic moduli, related to the flexural and torsional rigidities of equations (7) by

$$D_x = \frac{E_x t^3}{12}, \qquad D_y = \frac{E_y t^3}{12}, \qquad D_1 = \frac{E_{xy} t^3}{12}, \qquad D_{xy} = \frac{G_{xy} t^3}{12}$$

in which t is the slab thickness.

By using average stiffness values for any stiffened or composite slab or grid, the physical structure may be transformed into an equivalent orthotropic plate.

The strain energy due to bending and twisting of the plate then becomes

$$U = \frac{6cl}{\pi t^3}\int_0^\pi \int_{-1}^1 \{A_1 M_x^2 + A_2 M_y^2 + 2A_{12} M_x M_y + A_3 M_{xy}^2\}\, \mathrm{d}\eta\, \mathrm{d}\xi \tag{11}$$

in which

$$A_1 = \Delta E_y, \qquad A_2 = \Delta E_x, \qquad A_{12} = -\Delta E_{xy}$$

$$A_3 = \frac{1}{G_{xy}}, \qquad \text{and} \qquad \Delta = \frac{1}{E_x E_y - E_{xy}^2}.$$

The strain energy coefficients A, introduced here for simplicity, are given by

$$\left.\begin{aligned} &A_1 = kD_y, \qquad A_2 = kD_x, \qquad A_{12} = -kD_1, \\ \text{and} \qquad &A_3 = k\left(\frac{D_x D_y - D_1^2}{D_{xy}}\right), \qquad \text{where} \qquad k = \frac{12}{t^3}\left(\frac{1}{D_x D_y - D_1^2}\right). \end{aligned}\right\} \tag{12}$$

On substituting equations (10) into (11), followed by integration over the span of the slab, the strain-energy integral reduces to

$$\begin{aligned} U = \frac{3cl}{t^3}\int_{-1}^1 \Bigg\{ &A_1 \sum_{i=1}^{\infty} \left(r_i^2 \frac{\mathrm{d}^2 \mathrm{F}_i}{\mathrm{d}\xi^2} + 2r_i \frac{\mathrm{dR}_i}{\mathrm{d}\xi} + p_i c^2 r_i^2 \right)^2 \\ &+ A_2 \sum_{i=1}^{\infty} \mathrm{F}_i^2 + 2A_{12} \sum_{i=1}^{\infty} \mathrm{F}_i \left(r_i^2 \frac{\mathrm{d}^2 \mathrm{F}_i}{\mathrm{d}\xi^2} + 2r_i \frac{\mathrm{dR}_i}{\mathrm{d}\xi} + p_i c^2 r_i^2 \right) \\ &+ A_3 \left(2\mathrm{S}^2 + \sum_{i=1}^{\infty} \mathrm{R}_i^2 \right) \Bigg\} \mathrm{d}\xi. \end{aligned} \tag{13}$$

The strain energy in the plate must be a minimum. On minimizing by the calculus of variations, the necessary condition that the integral (13) is a minimum is that the functions F_i and R_i satisfy a set of ordinary linear differential equations of the form

$$(A_1 r_i^4 \mathrm{D}^4 + 2A_{12} r_i^2 \mathrm{D}^2 + A_2)\mathrm{F}_i + 2(A_1 r_i^3 \mathrm{D}^3 + A_{12} r_i \mathrm{D})\mathrm{R}_i = -(A_1 r_i^2 \mathrm{D}^2 + A_{12}) p_i c^2 r_i^2 \tag{14}$$

$$2(A_1 r_i^3 \mathrm{D}^3 + A_{12} r_i \mathrm{D})\mathrm{F}_i + (4A_1 r_i^2 \mathrm{D}^2 - A_3)\mathrm{R}_i = -2A_1 c^2 r_i^3 \mathrm{D} p_i \tag{15}$$

in which, for convenience, the operator $\mathrm{D} = \mathrm{d}/\mathrm{d}\xi$ is used, and i is any integer.

In addition, it is found that the function S vanishes.

The required boundary conditions follow naturally from the integrated terms, evaluated at the limits of the expansion, in the minimizing procedure, in accordance with the known physical edge conditions along each free edge (equations (5)). In this case, it is found that the necessary boundary conditions become, at $\xi = \pm 1$,

$$\mathrm{F}_i = r_i \frac{\mathrm{dF}_i}{\mathrm{d}\xi} + 2\mathrm{R}_i = 0. \tag{16}$$

Equations (16) express the conditions of vanishing normal bending moments and equivalent shear forces along the free edges of the slab.

Solution of equations (14) and (15)

The simplest form of solution of equations (14) and (15) is obtained by considering the applied load system as a superposition of symmetric and anti-symmetric components with respect to the central axis $0x$. In that case, for a symmetric load system, the functions F_i will be symmetric and the functions R_i anti-symmetric, whilst the converse will be true for an anti-symmetric load system. In each case, only two boundary conditions need be satisfied, instead of the four normally specified.

The solution of equations (14) and (15) will consist of two parts, the complementary function solution (independent of the load system), and the particular integral solution (which depends on the form of the applied load).

(a) *Complementary function solution*

For the range of relative stiffness values encountered in practical orthotropic slab structures, the roots of the auxiliary equation, corresponding to equations (14) and (15), will always be complex conjugate (except for the particular case of isotropic slabs, when repeated roots are obtained), and the complementary function solutions may always be expressed in the following forms:

(i) *Symmetric load system*

$$\begin{aligned}\mathrm{F}_i = {} & K_{1_i}\{B_1 \cosh S_i\xi \cos t_i\xi - B_2 \sinh S_i\xi \sin t_i\xi\} \\ & + K_{2_i}\frac{1}{t_i}\{B_1 \sinh S_i\xi \sin t_i\xi + B_2 \cosh S_i\xi \cos t_i\xi\}\end{aligned} \tag{17a}$$

$$\begin{aligned}\mathrm{R}_i = {} & K_{1_i}\{B_3 \sinh S_i\xi \cos t_i\xi + B_4 \cosh S_i\xi \sin t_i\xi\} \\ & + K_{2_i}\frac{1}{t_i}\{B_3 \cosh S_i\xi \sin t_i\xi - B_4 \sinh S_i\xi \cos t_i\xi\}\end{aligned} \tag{17b}$$

(ii) *Anti-symmetrical load system*

$$\begin{aligned}\mathrm{F}_i = {} & K_{3_i}\frac{1}{t_i}\{B_1 \cosh S_i\xi \sin t_i\xi + B_2 \sinh S_i\xi \cos t_i\xi\} \\ & + K_{4_i}\{B_1 \sinh S_i\xi \cos t_i\xi - B_2 \cosh S_i\xi \sin t_i\xi\}\end{aligned} \tag{18a}$$

$$\begin{aligned}\mathrm{R}_i = {} & K_{3_i}\frac{1}{t_i}\{B_3 \sinh S_i\xi \sin t_i\xi - B_4 \cosh S_i\xi \cos t_i\xi\} \\ & + K_{4_i}\{B_3 \cosh S_i\xi \cos t_i\xi + B_4 \sinh S_i\xi \sin t_i\xi\}\end{aligned} \tag{18b}$$

in which α and β are given by

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \sqrt{\left[\frac{1}{2}\left\{\left(\frac{D_x}{D_y}\right) \pm \frac{D_1+2D_{xy}}{D_y}\sqrt{\left(\frac{D_x}{D_y}\right)}\right\}\right]}$$

$$S_i = \alpha/r_i, \qquad t_i = \beta/r_i,$$

$$B_1 = 4A_1(\alpha^2-\beta^2)-A_3$$

$$B_2 = 8A_1\alpha\beta$$

$$B_3 = 2\{A_1(3\beta^2-\alpha^2)-A_{12}\}\alpha$$

$$B_4 = 2\{A_1(3\alpha^2-\beta^2)+A_{12}\}\beta.$$

In the particular case of an isotropic slab, the stiffnesses reduce to

$$D_x = D_y = D, \qquad D_1 = \nu D, \qquad D_{xy} = \frac{1-\nu}{2}D$$

where D is the flexural rigidity of the plate, equal to $Et^3/12(1-\nu^2)$, where E is Young's modulus and ν is Poisson's ratio for the material. In that case, repeated roots are obtained with $\alpha = 1$ and $\beta = 0$.

It is worth noting that repeated roots may also be obtained with orthotropic plates in which particular relationships occur between the different stiffnesses. Such a case occurs with the stiffness values suggested by Timoshenko and Woinowsky-Krieger [2] for reinforced concrete slabs with two-way reinforcement. If Dx and Dy are the equivalent flexural stiffnesses in the spanwise and transverse directions, the other stiffnesses are given approximately by

$$D_1 = \nu_c\sqrt{(DxDy)}, \qquad \text{and} \qquad D_{xy} = \frac{1-\nu_c}{2}\sqrt{(DxDy)}$$

where ν_c is Poisson's ratio for concrete.

Equations (17) and (18) have been expressed in such a form that they reduce to the required mathematical solutions in the limiting case of repeated roots.

(b) Particular integral solutions

The particular integral solutions will depend on the load system, and no general solution can be given. However, since the main problem in the analysis of bridge decks is the determination of the stress distribution due to a concentrated load applied at any given position on the slab, the particular integral is derived for this load form.

A concentrated load of magnitude P applied at any point (x_1, y_1) on the slab may be considered as a superposition of a symmetric system, consisting of loads of magnitudes $P/2$ at the positions (x_1, y_1) and $(x_1, -y_1)$, and an anti-symmetric system, consisting of a load $P/2$ at (x_1, y_1) and a load $-P/2$ at $(x_1, -y_1)$.

(i) *Symmetrical load system.* A symmetrical two-point load system may be expressed as a Fourier series of the form given in equation (8) as

$$p = \sum_{i=1}^{\infty} p_i \sin i\eta$$

where

$$p_i = \frac{P}{lc}\left\{1+2\sum_{j=1}^{\infty}\cos j\pi\xi_1\cos j\pi\xi\right\}\sin i\eta_1 \tag{19}$$

where the point loads, of magnitudes $P/2$, are applied at the positions (η_1, ξ_1) and $(\eta_1, -\xi_1)$, defined in terms of the non-dimensional co-ordinate system used.

Particular integral solutions of equations (14) and (15) may then be shown to be

$$\begin{aligned} \mathrm{F}_i &= Q_i\left\{2\sum_{j=1}^{\infty}\left(\frac{1}{\Delta_j}(g_j^2 - A_{12}/A_1)\cos j\pi\xi_1\cos j\pi\xi\right) - \frac{A_{12}}{A_2}\right\} \\ \mathrm{R}_i &= 4Q_i\left\{\sum_{j=1}^{\infty}\frac{1}{\Delta_j}\left(\frac{A_{12}^2}{A_1A_3} - \frac{A_2}{A_3}\right)g_j\cos j\pi\xi_1\sin j\pi\xi\right\} \end{aligned} \tag{20}$$

in which, for convenience, the following are used

$$g_j = j\pi r_i$$

$$Q_i = p\left(\frac{c}{l}\right)r_i^2\sin i\eta_1$$

$$\Delta_j = g_j^4 - 2\left(\frac{A_{12}}{A_1} - 2\frac{A_2}{A_3} + 2\frac{A_{12}^2}{A_1A_3}\right)g_j^2 + A_2/A_1.$$

(ii) *Anti-symmetrical load system.* The corresponding anti-symmetrical load system may be expressed as

$$p = \sum_{i=1}^{\infty} p_i \sin i\eta$$

where

$$p_i = \frac{2P}{lc}\sin i\eta_1\sum_{j=1}^{\infty}\sin j\pi\xi_1\sin j\pi\xi \tag{21}$$

the point loads, of magnitude $P/2$ and $-P/2$, being applied at the positions (η_1, ξ_1) and $(\eta_1, -\xi_1)$ respectively.

Particular integral solutions of equations (14) and (15) may then be expressed as

$$\begin{aligned} \mathrm{F}_i &= 2Q_i\left\{\Sigma\frac{1}{\Delta_j}(g_j^2 - A_{12}/A_1)\sin j\pi\xi_1\sin j\pi\xi\right\} \\ \mathrm{R}_i &= 4Q_i\left(\frac{A_2}{A_3} - \frac{A_{12}^2}{A_1A_3}\right)\left\{\Sigma\frac{g_j}{\Delta_j}\sin j\pi\xi_1\cos j\pi\xi.\right\} \end{aligned} \tag{22}$$

Evaluation of integration constants K

(i) *Symmetrical case.* On substituting the solutions for F_i and R_i from equations (17) and (20) into the boundary conditions (16), the constants K_{1_i} and K_{2_i} are found to be

$$K_{1_i} = \frac{K_{2_i}}{\theta_1}\frac{1}{t_i}\{\alpha\cosh S_i\sin t_i + \beta\sinh S_i\cos t_i\} \tag{23a}$$

$$K_{2_i} = Q_i\frac{\theta_1}{\theta_2}\left\{\frac{A_{12}}{A_1} - 2\Sigma\frac{(-1)^j}{\Delta_j}\left(g_j^2 - \frac{A_{12}}{A_1}\right)\cos j\pi\xi_1\right\} \tag{23b}$$

where

$$\theta_1 = \beta \cosh S_i \sin t_i - \alpha \sinh S_i \cos t_i$$

and

$$\theta_2 = \frac{1}{t_i}\{B_1(\alpha \sin t_i \cos t_i + \beta \sinh s_i \cosh s_i) - B_2(\alpha \cosh s_i \sinh s_i - \beta \sin t_i \cos t_i)\}.$$

(ii) *Anti-symmetrical case.* Substitution of the anti-symmetrical solutions (18) and (21) into equations (16) yields the constants K_{3_i} and K_{4_i}:

$$K_{3_i} = \frac{K_{4_i}}{\theta_3}\frac{\beta}{r_i}\{B_2 \cosh s_i \sin t_i - B \sinh s_i \cos t_i\}$$

$$K_{4_i} = \frac{2Q_i}{A_3+4A_{12}}\frac{\theta_3}{\theta_4}\left\{\sum_{j=1}^{\infty}\frac{g_j(-1)^j}{\Delta_j}\left(g_j^2 - \frac{A_{12}}{A_1} + \frac{4A_2}{A_3} - \frac{4A_{12}^2}{A_1 A_3}\right)\sin j\pi\xi_1\right\} \tag{24}$$

where

$$\theta_3 = B_1 \cosh s_i \sin t_i + B_2 \sinh s_i \cos t_i$$

$$\theta_4 = B_1(\alpha \sin t_i \cos t_i - \beta \sinh s_i \cosh t_i) + B_2(\alpha \sinh s_i \cosh s_i + \beta \sin t_i \cos t_i).$$

The complete solution for a single point load is then obtained by superposition of the symmetrical and anti-symmetrical systems.

The other commonly considered cases of a uniformly distributed load, and a line load parallel to the supports, are treated in Appendix 1.

The influence of shear deformations

The influence of shear deformations may be included in the analysis by adding the strain energy of shear to that of bending and twisting in equation (11). If Reissner's boundary conditions, equations (4), are then utilized, the function S in equations (10) must be expressed as

$$\mathrm{S} = -\sum_{i=1}^{\infty} \mathrm{R}_i(-1)^i.$$

The analysis may then be followed through as before. In this case, a sixth-order set of governing equations is obtained, enabling all three physical free-edge boundary conditions of equations (3) to be satisfied. However, because of the form of the function S above, the set of equations no longer falls into groups of two independent equations, and the complete set must be solved simultaneously. Consequently, it is not possible to obtain general solutions, and the analysis is not proceeded with any further here.

NUMERICAL EXAMPLE

The rate of convergence of the solution is examined through the typical example of an isotropic square bridge slab carrying a central point load. The low span/width ratio is chosen deliberately to produce a marked transverse stress variation, and provide a good test of the analysis. The variation of the central transverse moment M_y, as the number of terms in the solution is increased, is shown in Fig. 2. It was found that the

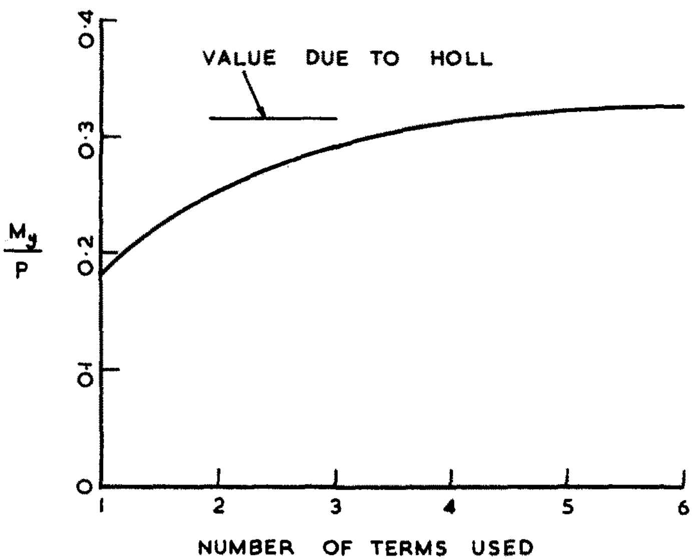

FIG. 2. Convergence of central transverse moment.

complementary function component was negligible except for the first term, so that computations were much simplified as only the particular integral component of the solution was required to be evaluated.

The value given by Holl [3], obtained by a difference method, is indicated for comparison.

DISCUSSION

A method has been presented for the direct determination of the bending and twisting moments, and shear forces, in right orthotropic bridge decks.

The two-dimensional partial differential equation of plate theory is rendered unidirectional by the assumption that the stress-resultants may be expressed as Fourier series in the spanwise co-ordinate, the coefficients of the series being functions of the transverse position only. The problem reduces to the solution of a set of ordinary linear differential equations, in groups of two, each containing only two functions, enabling solutions to be obtained in general terms as the summation of infinite series. As the series converge fairly rapidly, a few terms only are sufficient to yield accurate results.

Although the equations produced are lengthy, they are simpler than those obtained by Massonnet, by virtue of the fact that the solution is obtained as a superposition of symmetrical and anti-symmetrical systems. The solution of the governing equations in the analysis yields directly the transverse bending moments and twisting moments, as a set of stress resultant functions F_i and R_i. The spanwise moments are obtained by differentiation of these functions, and hence converge less rapidly; consequently, the method is most suitable for the determination of the moments M_y and M_{xy}. This is of interest, as the conventional method of analysis is more suitable for the determination of the spanwise moments, M_x.

The results may be modified considerably if certain simplifying assumptions are made. With reinforced concrete structures, it is frequently assumed that the value of Poisson's ratio is effectively zero, in which case the coefficient A_{12} vanishes. The results may also be applied to the analysis of grillage structures, in which it is often assumed that no torsion occurs, and vertical forces only are transmitted between the longitudinal and transverse members. In that case, there are no twisting moments, and hence the functions R and S vanish in equations (10), and equation (15) is no longer applicable.

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APPENDIX 1

Solutions for other load cases

The complementary function solutions given in equations (17) and (18) depend only on the stiffness and geometrical properties of the slab, and are independent of the applied load system. Consequently, only the particular integral solutions (20) and (22) need be altered to cater for other loading cases.

Two further cases are considered here, the uniformly distributed load, and the line load applied parallel to the supports, both symmetrical systems.

(a) *Uniformly distributed load.* If the slab is subjected to a uniformly distributed load of intensity p_0, particular integral solutions of equations (14) and (15) may be expressed as

$$\begin{aligned} \mathrm{F}_i &= -\frac{4A_{12}}{A_2} p_0 \frac{c^2 r_i^2}{i\pi} \sin^2\left(\frac{i\pi}{2}\right) \\ \mathrm{R}_i &= 0 \end{aligned} \tag{25}$$

(b) *Line load.* If a line load of total magnitude P is applied parallel to the supports at any station η_1, particular integral solutions are

$$\begin{aligned} \mathrm{F}_i &= -\frac{A_{12}}{A_2} P\left(\frac{c}{l}\right) r_i^2 \sin i\eta_1 \\ \mathrm{R}_i &= 0 \end{aligned} \tag{26}$$

Evaluation of constants K

On satisfying the boundary conditions at the free edges of the plate, the constants K_{1_i} and K_{2_i} are again related by equation (23a). Equation (23b) must be replaced by

$$K_{2_i} = \frac{\theta_1}{\theta_2} \mathrm{F}_i$$

where F_i must be replaced by the appropriate expression in equations (25) or (26).

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Résumé—La détermination directe des moments de flexion et de torsion, et des forces de cisaillement des tabliers de ponts orthotropes, est étudiée dans cet article. L'équation aux dérivées partielles de la théorie des plaques est ramenée à une série d'équations différentielles ordinaires du premier degré, en groupe de deux, chaque groupe ne comprenant que deux fonctions inconnues, en supposant que les résultantes de contrainte peuvent s'exprimer en suites de Fourier, en coordonnée dans le sens de la travée, les coefficients de la suite étant fonction de la position transversale seulement. Les suites sont choisies pour satisfaire les conditions d'équilibre de la plaque, le coefficient inconnu étant obtenu par le Principe du Moindre Effort.

Zusammenfassung—Die direkte Bestimmung der Biegungs und Drehmomente und der Scherkräfte in orthotropen Brückenfahrbahnen ist in dieser Abhandlung erwogen. Die partielle Differentialgleichung der Plattentheorie ist zu einem Satz von gewöhnlichen linearen Differentialgleichungen reduziert in Gruppen von zwei, woven jede Gruppe nur zwei unbekannte Funktionen enthält, in der Annahme das die Beanspruchungsresultanten als Fouriersche Reihen in spannweisen Koordinaten ausgedrückt werden können, die Koeffizienten der Reihen sind nur Funktionen der Querlage. Die Reihen wurden gewählt um die Gleichgewichtsbedingungen zu befriedigen, die unbekannten Koeffizienten werden durch das Gesetz der Geringsten Arbeit erhalten.

Абстракт—В этой статье обсуждается прямое определение моментов сгибания, кручения и срезывающих сил в ортотропических мостовых настилах. Частное дифференциальное уравнение теории пластины уменьшено до подбора обыкновенных линейных дифференциальных уравнений, в группах по два, каждая группа содержит только две неизвестных функции, предполагая, что равнодействующие напряжения могут быть выражены, как серии Фурье (Fourier) в дугообразной координате, и коэффициенты серий представляют функции только поперечной позиции. Серии выбраны для удовлетворения условий равновесия для пластины, неизвестные коэффициенты получены по принципу наименьшей работы.